LITERATURE CITED

1. S. G. Mikhlin, Variational Methods in Mathematical Physics [in Russian], Nauka, Moscow (1970).
2. M. A. Lavrent'ev, Variational Methods in Boundary-Value Problems for Systems of Equations of Elliptic Type [in Russian], Izd. Akad. Nauk SSSR, Moscow (1962).
3. D'yarmati, Nonequilibrium Thermodynamics. Theory of Fields and Variational Principles [Russian translation], Mir, Moscow (1974).
4. V. V. Denisenko, "Symmetric form of elliptic equations describing transport processes in gyrotropic media," Deposited paper No. 4896-80, VINITI, November 5, 1980, Krasnoyarsk (1980).
5. V. V. Denisenko, "Boundary-value problem for a symmetric strongly elliptic system under a changing type of boundary condition," Deposited paper No. 1008-81, VINITI, March 4, 1981, Krasnoyarsk (1981).
6. V. V. Denisenko, "Symmetric form of three-dimensional elliptic equations describing transport processes in gyrotropic media," Deposited paper No. 344-81, VINITI, January 28, 1981, Krasnoyarsk (1981).
7. V. V. Denisenko, "Energetic method of solving three-dimensional mixed boundary-value problems for an elliptic equation with nonsymmetric coefficient matrix," Deposited paper No. 5647-81, VINITI, December 14, 1981, Krasnoyarsk (1981).
8. S. L. Sobolev, Certain Applications of Functional Analysis in Mathematical Physics [in Russian], Izd. Sib. Otd. Akad. Nauk SSSR, Novosibirsk (1962).
9. A. V. Bitsadze, Principles of the Theory of Analytic Complex Functions [in Russian], Nauka, Moscow (1972).
10. V. V. Denisenko and S. S. Zamai, "Symmetrization and numerical solution of elliptic boundary-value problems with nonself-adjoint operators," in: Modeling of Hydrogasdynamics and Energetics Problems [in Russian], Inst. Teor. Prikl. Mekh., Sib. Otd. Akad. Nauk SSSR, Novosibirsk (1984).
11. V. V. Denisenko and S. S. Zamai, "Numerical modeling of electrical fields and currents in a low-latitude ionosphere," in: Mathematical Models and Methods of Solution of Problems of Mechanics of a Continuous Medium [in Russian], Vychisl. Tsentr, Sib. Otd. Akad. Nauk SSSR, Krasnoyarsk (1986).

STABILITY OF THE SURFACE OF A GAS BUBBLE PULSATING IN A LIQUID

O. V. Voinov and V. V. Perepelkin

UDC 541.24:532.5

This article examines the stability of the surface of a spherical gas bubble undergoing nonlinear oscillations. We study the dynamics of small perturbations as a function of the wavelength and parameters of the nonlinear bubble pulsations. An approach is developed for analyzing the dynamics of the bubble-surface perturbations on the basis of solution of the differential equation of stability for a pulsation half-period. The shortwave approximation is used to obtain a formula for the increment of the perturbation, and an analogy is established between the stability problem and the problem of the passage of a particle across a potential barrier in quantum mechanics. Asymptotic formulas are found for the rate of growth of perturbations in the case of large-amplitude pulsations, and a comparison is made with exact numerical calculations. It is shown that the rate of growth of perturbations of a prescribed wavelength is a bounded function with infinite intensification of the pulsations. With consideration of capillary forces, it was found that the most rapidly growing perturbations shift in the shortwave direction as the amplitude of the pulsations intensifies. It is shown that Taylor instability is the main reason for rupture of the surface of the pulsating gas bubble.

The stability of a plane interface between two liquids was first examined by Taylor [1] in connection with the problem of bubble dynamics in an underwater explosion. Experiments

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 76-83, May-June, 1989. Original article submitted August 18, 1987.

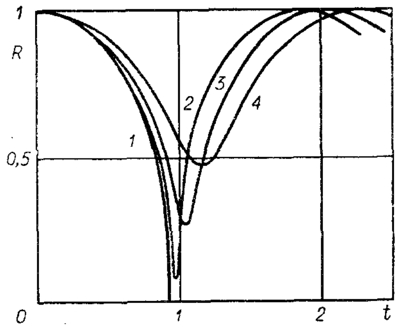

Fig. 1

revealed a needlelike disturbance of the spherical shape of the bubble, which took the form of a "sea urchin" [2]. Taylor showed that the interface was unstable if the acceleration of the boundary was directed from the less-dense toward the more-dense liquid. The authors of [3] studied the instability of a spherical boundary of an expanding or collapsing cavitational cavity in the case where the acceleration of the cavity wall was sign-constant and was directed toward the cavity's center. Such a boundary is stable in Taylor's theory but, due to the spherical symmetry of the problem, the amplitudes of the perturbations of the cavity surface depend on the radius of the bubble R as $R^{-1/4}$. Here, the collapsing cavity turns out to be unstable (Birkhoff–Plesset instability). The stability of the surface of a periodically pulsating gas bubble has been studied only for linear or slightly nonlinear pulsations [4], when the radius R(t) is described well by the first three terms of a Fourier series. Parametric excitation of surface waves was studied on the basis of the stability equation, which in this case takes the form of a Mathieu equation. The method used in [4] is inapplicable for large-amplitude pulsations, when the graph of the function R(t) has sharp discontinuities.

1. Formulation of the Problem and Method of Solution. In the absence of body forces, a gas bubble of radius R_0' is at rest in an infinite volume of an ideal incompressible fluid. At a certain moment of time $t' = 0$, the pressure in the liquid at infinity p_0 suddenly changes to p_∞. As a result, the bubble begins to pulsate. It is assumed that the pressure of the gas inside the bubble is described by the polytropic curve $p_g = \text{const}\,\rho_g^k$ (k is the index of the curve) and that the density of the gas ρ_g inside the bubble is always much less than the density of the surrounding fluid ρ_ℓ: $\rho_g \ll \rho_\ell$.

Let the following equation describe the surface of the bubble [3] in a spherical coordinate system r, θ, φ with its origin at the bubble's center

$$r = R' + \sum_{n=2}^{\infty} a_n' Y_n(\theta, \varphi), \quad |a_n'| \ll R',$$

where $R'(t')$ is the running radius of the bubble; $a_n'(t')$ is the small amplitude of the deviation from spherical form; $Y_n(\theta, \varphi)$ is a spherical surface harmonic of degree n. Then the equations for R' and a_n' can be written in dimensionless form

$$\ddot{R}R + \frac{3}{2}\dot{R}^2 = (\varepsilon + 2\sigma)R^{-3k} - \frac{2\sigma}{R} - 1, \quad R(0) = 1, \ \dot{R}(0) = 0; \tag{1.1}$$

$$\ddot{y}_n - \Phi_n(t)\,y_n = 0, \quad \Phi_n(t) = \left(n + \frac{1}{2}\right)\frac{\ddot{R}}{R} + \frac{3}{4}\left(\frac{\dot{R}}{R}\right)^2 - (n^2 - 1)(n + 2)\frac{\sigma}{R^3}. \tag{1.2}$$

Here $R = R'/R_0'$, $y_n = a_n' R^{3/2}/R_0'$, $t = t'(p_\infty/\rho_\ell)^{1/2}/R_0'$, $\varepsilon = p_0/p_\infty$; $\sigma = \sigma'/R_0' p_\infty$; t' and σ' are the dimensionless time and surface tension; the dot above the letters denotes differentiation with respect to t. Equations (1.2) for modified amplitudes y_n with different numbers n are mutually independent, so we will henceforth omit the subscript n.

Figure 1 shows characteristic graphs of the solutions of Eq. (1.1) with different ε and σ = 0. Curves 1-4 correspond to ε = 0, 0.02, 0.1, and 0.25. The functions R(t) and thus Φ(t), are periodic. Their period corresponds to the pulsations of the bubble T. Thus, (1.2) is an equation of the Hill type. In accordance with Floke's theory [5], its solution has the form $y = \xi(t)\exp(\mu_1 t) + \zeta(t)\exp(\mu_2 t)$, where ξ and ζ are T-periodic functions. Since Φ(t) is even, then $\mu_1 = -\mu_2 \equiv \mu$, and the characteristic index $\mu = \mathrm{Ln}(y_1(T) \pm \sqrt{y_1^2(T) - 1})/T$.

Here $y_1 = y_1(T)$ is the solution of Eq. (1.2) with the initial conditions $y_1(0) = 1$, $\dot{y}_1(0) = 0$. The spherical surface of the bubble will be stable against perturbations with the number n if the amplitude of $y(t, n, \varepsilon, \sigma, k)$ is finite over time. For this to occur, the characteristic index μ must be a purely imaginary number. The latter is true only when $|y_1(T)| < 1$. At $|y_1(T)| > 1$,

$$\operatorname{Re}\mu = \ln\left(|y_1(T)| + \sqrt{y_1^2(T) - 1}\right)/T \tag{1.3}$$

and the amplitude of the perturbations increases exponentially. Thus, to solve the stability problem and determine the law governing perturbation growth, it is sufficient to calculate the value of $y_1(T)$. Let $Y_0(t) = \begin{pmatrix} y_1 & \dot{y}_1 \\ y_2 & \dot{y}_2 \end{pmatrix}$ be the matrix of two linearly independent solutions of Eq. (1.2) with unit initial data $Y_0(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then $Y(T/2) = Y(0)D$, $Y(T/2)D^{-1} = Y(0)$ $[D = Y_0(T/2)]$ is valid for the matrix Y of two arbitrary linearly independent solutions with the initial values Y(0). Considering the symmetry of the functions R(t) and $\Phi(t)$ relative to the moment of time $t = T/2$, and making the substitution $t \to T - t$ in the last formula, with $\dot{y}$ becoming $-\dot{y}$ in this case, we obtain the following:

$$Y(T) = Y(T/2)JD^{-1}J = Y(0)DJD^{-1}J, \quad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In particular, $Y_0(T) = DJD^{-1}J$. By virtue of the fact that $|D| = 1$, it follows that

$$y_1(T) = 2y_1(T/2)\,\dot{y}_2(T/2) - 1, \tag{1.4}$$

and to solve the stability problem it is sufficient to calculate Eq. (1.2) for a pulsation half-period.

2. Dynamics of Shortwave Perturbations ($n \gg 1$). The equation for the amplitudes of the perturbations (1.2) can be rewritten in the form

$$\ddot{y} + \left(\lambda^2 q(t) - \frac{3}{4}\left(\frac{\dot{R}}{R}\right)^2\right) y = 0; \tag{2.1}$$

$$q(t) = -\frac{\ddot{R}}{R} + \frac{2\sigma(n^2 - 1)(n + 2)}{R^3(2n + 1)}, \quad \lambda^2 = n + \frac{1}{2}. \tag{2.2}$$

The parameter σ is very small for sufficiently large bubbles. Thus, $\sigma \approx 0.15 \cdot 10^{-3}$ for a bubble in water with $R_0' = 10^{-3}$ m, $p_0 = 0.1$ MPa, and $\varepsilon = 0.2$. Thus, with nonlinear pulsations, when the acceleration of the wall reaches large values, the form of the function q(t) is determined within a broad range of n mainly by the term $-\ddot{R}/R$. As a result, the function q(t) will have a simple zero at the point $t_* < T/2$: $q(t_*) = 0$. Meanwhile, $q(t) > 0$ at $0 \le t < t_*$ and $q(t) < 0$ at $t_* < t \le T/2$. In the case $n \gg 1$, Eq. (2.1) can be regarded as an equation with a large parameter having a turning point.

In accordance with the WKB method, the two linearly independent solutions (2.1) can be written in the following form with the accuracy $O(\lambda^{-1})$ at $0 \le t < t_*$:

$$y_i = q^{-1/4} C_i^- \cos(\lambda I(t) + \pi/4 + \psi_i), \quad I(t) = \int_t^{t_*} \sqrt{q}\, dt. \tag{2.3}$$

The amplitudes of the perturbations remain finite on this time interval for any n. Similarly, the following is valid in the region $t_* < t \le T/2$ to within $O(\lambda^{-1})$:

$$y_i = (-q)^{-1/4} C_i^+ \exp(\lambda K(t)), \quad K(t) = \int_{t_*}^{t} \sqrt{-q}\, dt. \tag{2.4}$$

Here the exponentially small term $\exp(-\lambda K)$ is not considered. The exponentially increasing solution (2.4) corresponds to Taylor instability of the interface of two fluids during acceleration directed from the lighter to the heavier fluid.

The constants C_i^+ and C_i^- in Eqs. (2.3) and (2.4) are connected to each other by the solution of Eq. (2.1) in a small neighborhood of the turning point t_*, where asymptotic representations (2.3) and (2.4) lose significance. The problem of the transition in the asymptotic solution through a simple zero of the function q(t) has been solved in quantum mechanics in the quasiclassical theory of the passage of a particle across a potential barrier. In accordance with the conjugate formulas of quantum mechanics [6], the following is valid to within small $O(\lambda^{-2/3})$:

$$C_i^+ = C_i^- \cos \psi_i. \tag{2.5}$$

With allowance for the unit initial conditions for y_1, y_2, and $\dot{q}(0) = 0$, we can use Eqs. (2.3) (2.5) to find the corresponding values of the constants ψ_i, C_i^-: $\psi_1 + \lambda I(0) + \pi/4 = 0$, $C_1^- = q^{1/4}(0)$, $\psi_2 = \psi_1 + \pi/2$, $C_2^- = \lambda^{-1}q^{-1/4}(0)$. It follows from this and from (1.4), (2.4), and (2.5) that

$$y_1(T) = \exp(2\lambda K_0) \cos 2\lambda I_0,\ K_0 = K(T/2),\ I_0 = I(0). \tag{2.6}$$

With an increase in the number of the surface mode n, the characteristic index (1.3) increases in order of magnitude as $2\lambda K_0/T \sim \sqrt{n}$. This is true within a limited range of n in which capillary forces can be ignored.

3. Asymptotes with Nonlinear Pulsations ($\varepsilon \ll 1$). Let us examine the case when the capillary forces are insignificant and we can put $\sigma = 0$. Taking the first integral of the Rayleigh equation into account,

$$\dot{R}^2 = \frac{2}{3}\left((1+\alpha) R^{-3} - \alpha R^{-3k} - 1\right), \quad \alpha = \frac{\varepsilon}{k-1}, \tag{3.1}$$

we represent the function q(t) in (2.1) in the form

$$q(t) = -\ddot{R}/R = (1+\alpha)R^{-5} - k\alpha R^{-3k-2}. \tag{3.2}$$

In the limiting case $\varepsilon = 0$, Eqs. (2.1) and (3.1) and the above function q(t) describe the closure of a cavitational cavity in an infinite volume of an ideal fluid in the absence of capillary forces. During the final stage of closure $R \to 0$ the solutions of (2.1) are represented by the asymptote

$$y \sim \text{const}\, R^{5/4} \exp\left(i \sqrt{\frac{3}{2}\left(n - \frac{25}{24}\right)} \ln R\right), \quad i = \sqrt{-1}. \tag{3.3}$$

On the other hand, it follows from (2.3) at $n \gg 1$ that $y \sim \text{const}\, R^{5/4} \exp\left(i \frac{\lambda}{\sqrt{6}} \times \ln \frac{1+\sqrt{1-R^3}}{1-\sqrt{1-R^3}}\right)$.

At $R \to 0$, this expression has the asymptote, and for it to coincide exactly with (3.3) it is necessary to set $\lambda = \sqrt{n - 25/24}$. The parameter λ will henceforth be determined in this manner, not as it was in (2.2).

At $\varepsilon \to 0$, the radius of the bubble during pulsations changes within the range from its maximum value R(0) = 1 to the minimum value $R_m \equiv R(T/2) \to 0$:

$$R_m = b^{-1}(1 + A/(3k-3) + O(\alpha^{2/(k-1)})),\ A = (1+\alpha)^{-1}b^{-3},$$
$$b = (1 + 1/\alpha)^{1/(3(k-1))}.$$

At the moment of time $t_* \in (0, T/2)$, being a simple zero of the function q(t) (3.2), $R_* \equiv R(t_*) = b^{-1}k^{1/(3(k-1))}$. It should be noted that at $\varepsilon \ll 1$, $R_* \sim R_m \ll 1$. Equations (2.3), (2.6), and (3.2) lead us to

$$I_0 = \int_0^{t_*} \sqrt{q}\, dt = -\sqrt{\frac{3}{2}} \int_1^{R_*} L(R) \frac{dR}{R}, \tag{3.4}$$

$$L(R) = \sqrt{\frac{1+\alpha-\alpha k R^{-3(k-1)}}{1+\alpha-\alpha R^{-3(k-1)} - R^3}}.$$

The minus sign in front of the last integral is due to the fact that the velocity of the bubble wall is negative during the collapse stage: $\dot{R} \leq 0$.

For $R \sim R_*$ we can put

$$L(R) \approx L_1(R) = \sqrt{\frac{1+\alpha-\alpha k R^{-3(k-1)}}{1+\alpha-\alpha R^{-3(k-1)}}} \quad (3.5)$$

to within the terms $O(\alpha^{1/(k-1)})$. At $R \sim 1$, the following is valid to within small $O(\alpha^2)$:

$$L \approx L^+ = L|_{\alpha=0} + \left.\frac{\partial L}{\partial \alpha}\right|_{\alpha=0} \alpha, \quad L_1 \approx L^- = 1 + \left.\frac{\partial L_1}{\partial \alpha}\right|_{\alpha=0} \alpha. \quad (3.6)$$

Considering that $L^+ \to L^-$ at $R \to R_*$, for $\alpha \to 0$ we can prove the validity of a compound expansion suitable for any $R \in [R_*, 1]$: $L = L_1 + L^+ - L^- + O(\alpha^{1/(k-1)}, \alpha^2)$. It follows from this and from (3.5), (3.6) that $L = L_1 + L_2 + \alpha L_3/2 + O(\alpha^{1/(k-1)}, \alpha^2)$, $L_2 = (1 - R^3)^{-1/2} - 1$, $L_3 = R^{-3(k-1)}[(1 - R^{3k})(1 - R^3)^{-3/2} - 1 - kL_2]$.

Inserting these expressions into (3.4), we obtain the following to within the terms $\ln\alpha O(\alpha^{1/(k-1)}, \alpha^2)$:

$$I_0 = \sqrt{\frac{3}{2}}\left(I_1 + I_2 + \frac{\alpha}{2} I_3\right), \quad I_i = \int\limits_{R_*}^{1} L_i \frac{dR}{R}. \quad (3.7)$$

The first two integrals are calculated in elementary functions

$$I_1 = -\frac{1}{3(k-1)} \ln\left[\alpha \frac{k-1}{4}\left(\frac{\sqrt{k}+1}{\sqrt{k}-1}\right)^{\sqrt{k}}\right] + \alpha \frac{k+1}{6(k-1)} + O(\alpha^2), \quad I_2 = $$
$$= \frac{2}{3}\ln 2 + O\left(\alpha^{1/(k-1)}\right).$$

Within the scope of the accuracy of our calculation of I_0, the integral $I_3 = \int\limits_0^1 L_3 \frac{dR}{R} + O\left(\alpha^{1/(k-1)}\right)$ depends parametrically only on k and can be easily calculated numerically. Analogously to I_0, the value of K_0 is also determined approximately. In accordance with (2.4), (2.6), and (3.2),

$$K_0 = \int\limits_{t_*}^{T/2} \sqrt{-q}\, dt = \frac{1}{\sqrt{6}(k-1)} \int\limits_{x_*}^{x_m} M(x) \frac{dx}{x},$$
$$x = \frac{\alpha}{1+\alpha} R^{-3(k-1)}, \; x_* = k^{-1}, \; x_m = 1 + O\left(\alpha^{1/(k-1)}\right), \quad (3.8)$$
$$M = \sqrt{\frac{kx-1}{1-x-Ax^{-1/(k-1)}}} = \sqrt{\frac{kx-1}{1-x}} + O\left(\alpha^{1/(k-1)}\right).$$

Integrating the approximate value of M within the range from x_* to 1, to within $\ln\alpha O(\alpha^{1/(k-1)})$ we obtain

$$K_0 = \pi/[\sqrt{6}(\sqrt{k}+1)]. \quad (3.9)$$

Table 1 compares for different k exact values of K_0 determined on a computer from Eq. (3.8) and approximate values of the same from (3.9). The top number is the exact value calculated with $\varepsilon = 0.1$, while the bottom number is the approximate value.

Table 2 shows values of μ for $k = 1.4$, calculated from the same approximate formulas (bottom number) and found from the amplitude of $y_1(T)$ as determined by numerical calculation of system (1.1)-(1.2) (top number). The appreciable divergence of the values of μ at $\varepsilon = 0.1$ and $n = 50$ can be attributed to the fact that the argument of the cosine in (2.6) is calculated by means of (3.7) to within the terms $\sqrt{n}\ln\alpha O(\alpha^{1/(k-1)}, \alpha^2)$. Thus, for more accurate determination of the characteristic index with large n and moderately large ε, it is necessary

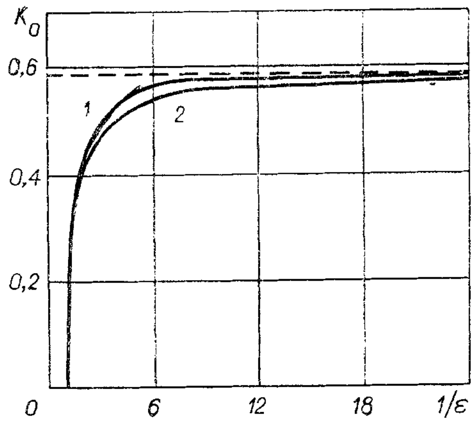

Fig. 2

TABLE 1

k	K_0
1,1	0,625 0,626
1,2	0,609 0,612
1,3	0,593 0,599
1,4	0,577 0,587

TABLE 2

n	μ		
	ε		
	0,1	0,05	0,01
2	0,992 0,882	0,314 0,272	0,414 0,602
6	1,251 1,109	1,675 1,650	1,544 1,546
10	2,003 1,949	2,051 2,027	2,271 2,251
50	4,069 2,654	4,512 4,557	4,132 4,110

to calculate I_0 directly from (3.4) by numerical integration. It should be noted that approximate calculation of K_0 from (3.9) does not yield a large relative error in the determination of μ.

Equation (3.9) shows that at $\varepsilon \to 0$ there is a limiting value of K_0. Curve 1 in Fig. 2 shows the graph of the dependence of K_0 (3.8) at $k = 1.4$ on the pressure gradient $p_\infty/p_0 = 1/\varepsilon$. The dashed line corresponds to the limiting value (3.9). It is evident that K_0 depends significantly on ε only in the region of relatively small pulsation amplitudes. In this region, when the linear theory of bubble pulsations is valid, $(1 - \varepsilon \ll 1)$, we have $R(t) = 1 - (1 - \varepsilon)(1 - \cos\omega t)/\omega^2$, $\omega^2 = 3k$, and $K_0 = 0.692\sqrt{(1 - \varepsilon)/k}$. In particular, for $k = 1.4$, $K_0 = 0.585\sqrt{1 - \varepsilon}$. Curve 2 shows the graph of the values of K_0 obtained from this formula. It is noteworthy that the formula obtained on the basis of the linear theory accurately describes the behavior of K_0 in the region of large nonlinear pulsations of the bubble.

Given sufficiently small initial perturbations, the main role is played by perturbations with a maximum characteristic index corresponding to the number of the mode $n = N$. The number N can be evaluated on the basis of consideration of the capillary forces. In the shortwave region $(n \gg 1)$, viscosity may also be important. For simplicity, this is not considered here.

4. Effect of Capillary Forces. With allowance for capillary forces, $\sigma \neq 0$. We will examine the case of a sufficiently large bubble, when $\sigma \ll 1$ and surface tension has an effect only in the shortwave region. The effect of σ on the behavior of the radius $R(t)$ can thus be ignored. The parameter K_0 in (2.4), (2.6), determining mainly the value of the characteristic index, will be written in the form

$$K_0 = \frac{1}{\sqrt{6}(k-1)} \int\limits_{x_*}^{x_m} \sqrt{\frac{kx - 1 - \Sigma x^{-2/3(k-1)}}{1 - x - Ax^{-1/(k-1)}}} \frac{dx}{x},$$

where x and x_m are determined in (3.8); $\Sigma \equiv \sigma n^2 b^{-2}(1 + \alpha)$; x_* is found from the equation $\Sigma = (kx_* - 1)x_*^{2/3(k-1)}$.

As shown above, in the case of nonlinear pulsations of large amplitude, we can set x_m = 1 and we can ignore the term containing $A \sim \alpha^{1/(k-1)}$:

$$K_0 \approx \frac{1}{\sqrt{6}(k-1)} \int_{x_*}^{1} \sqrt{\frac{kx - 1 - \Sigma x^{-2/3(k-1)}}{1-x}} \frac{dx}{x}.$$

An analytic approximation of this integral is possible if we make the substitution $kx - 1 - \Sigma x^{-2/3(k-1)} \approx (k - 1 - \Sigma)[(x - x_*)/(1 - x_*)]$. The error of calculation of the integral is no greater than 1% for k = 1.4. As a result,

$$K_0 \approx \frac{\pi}{\sqrt{6}(k-1)} \sqrt{(k-1-\Sigma)\frac{1-\sqrt{x_*}}{1+\sqrt{x_*}}}.$$

Since $\cos 2\lambda I_0$ in (2.6) is arapidly oscillating function of λ, the highest rate of growth will be seen for a perturbation with a number close to the number for which the index of the exponent $2\lambda K_0$ is maximal. The maximum of $2\lambda K_0$ is reached at a fixed value of Σ_N dependent parametrically only on k. Here, the number of the fastest-growing mode

$$N \approx \sqrt{\frac{\Sigma_N}{\sigma(1+\alpha)}}\, b, \quad b \approx \frac{1}{R_m}. \tag{4.1}$$

For k = 1.4, the maximum of $2\lambda K_0$ is reached at $\Sigma_N \approx 0.064$, $x_* \approx 0.783$, and

$$\max(2\lambda K_0) \approx 0.462\sqrt{b}(\sigma(1+\alpha))^{-1/4}. \tag{4.2}$$

An increase in n (n > N) is accompanied by an increase in the role of surface tension. The index $2\lambda K_0$ decreases and, beginning with a certain number N_*, capillary forces should suppress the development of Taylor instability at long wavelengths corresponding to $n > N_*$. With an increase in n, the turning point $t_* \to T/2$ and $K_0 \to 0$ in (2.4). Also, although a high degree of accuracy cannot be expected of the WKB method in the region $t_* < t \leq T/2$ either, it is nonetheless possible to evaluate N_* as the number of the mode at which $t_* = T/2$ and the region of negative values of q(t) in (2.2) disappears: $N_* \sim \sqrt{\frac{k-1}{\sigma(1+\alpha)}}\, b$.

Then we determine the region of numbers n which is most dangerous for breakup of the bubble as $N \leq n < N_*$. It should be noted that $N_*/N = \sqrt{(k-1)/\Sigma_N}$ and, in particular, $N_* \approx 2.5N$ for k = 1.4. We can use (4.1) and (4.2) to analyze the effect of the dimensionless pressure gradient ε, initial pressure p_0, initial bubble radius R_0', and surface tension σ' on the development of instability. It is evident that a change in these parameters leading to a reduction in ε or $\sigma = \varepsilon\sigma'/p_0R_0'$ increases N and the rate of growth of the corresponding perturbation, thus increasing the instability of the bubble surface.

It is interesting to evaluate the above quantities for a specific example. Let $R_0' = 10^{-3}$ m, p_0 = 0.1 MPa, ε = 0.1, σ' = 0.073 N/m, then $\sigma = 7.3\cdot10^{-5}$, $b \approx 3.8$, and, in accordance with (4.1), (4.2),

$$N \approx \sqrt{\frac{0.064}{\sigma(1+\alpha)}}\, b \approx 100, \quad \max(2\lambda K_0) \approx 9.24.$$

Thus, within two cycles of oscillation of the bubble radius, the amplitude of the most dangerous perturbation, with the number N, increases a huge number of times ($\sim10^8$). This is more than sufficient for breakup of the surface at this wavelength, if by breakup we mean the situation whereby the amplitude exceeds one-quarter of the wavelength, i.e., the small perturbation becomes a finite perturbation. It should be noted that the amplitude of perturbations with n ~ 2-4 simultaneously changes by only one or two orders of magnitude.

Thus, Taylor instability of the surface of the gas bubble is a more important factor than Birkhoff–Plesset instability caused by the spherical symmetry of the problem in the collapse of a cavitation bubble [3]. The latter instability does not intensify with repetition of the collapse cycle, while in the case of Taylor instability there is an increase in the amplitude of the perturbations in each cycle by the factor $\exp(2\lambda K_0)$. Moreover, the bubble must collapse many times for the manifestation of Birkhoff–Plesset instability, while the occurrence of Taylor instability requires 3-4 collapses over the radius if the wavelength of the disturbance is sufficiently small. It follows from this that the main reason for breakup of the surface of a pulsating gas bubble is Taylor instability.

We did not examine the case of a bubble which is simultaneously undergoing pulsations and translation. Thus, Kelvin–Helmholtz instability of the bubble surface (connected with discontinuity of tangential velocity at the gas–liquid boundary) remains outside the scope of the present discussion. However, it can be suggested that, given sufficiently small initial translation velocities of a bubble, Kelvin–Helmholtz instability will be less important than Taylor instability and the conclusions reached here will remain valid.

LITERATURE CITED

1. G. Taylor, "The instability of liquid surfaces when accelerated in a direction perpendicular to their planes," Proc. R. Soc., A201, 192 (1950).
2. R. Cowle, Underwater Explosions [Russian translation], IL, Moscow (1950).
3. M. S. Plesset and T. P. Mitchell, "On the stability of the spherical shape of a vapor cavity in a liquid," Q. Appl. Math., 13, No. 4 (1956).
4. W. Hermans, "On the instability of a translating gas bubble under the influence of a pressure step," Philips Res. Rep. Suppl., No. 3 (1973).
5. B. P. Demidovich, Lectures on the Mathematical Theory of Stability [in Russian], Nauka, Moscow (1967).
6. L. Schiff, Quantum Mechanics, McGraw-Hill, New York (1968).

STUDY OF NONSTEADY LOADS IN THE ACCELERATED AND SUDDEN MOTION OF BODIES OF DIFFERENT FORM

V. V. Podlubnyi and A. S. Fonarev

UDC 533.6.011

Together with the need to calculate the aerodynamic and strength characteristics of bodies during steady-state motion, it is often necessary to evaluate nonsteady forces acting during abrupt changes in the velocity regime – especially during sudden acceleration of a body from a state of rest to a specified steady flight velocity. It is interesting to determine the additional loads (compared to the steady phase of motion) that develop during nonsteady flow past the body. Here, the important characteristics are the maximum possible pressure and force and the characteristic time of the nonsteady transitional processes.

Below we examine the problem of the accelerated motion of certain bodies (a sphere, a cylinder with a flat edge, and a cone) from a state of rest to a specified subsonic or supersonic velocity with different accelerations. We will include the case of sudden motion of the body with a prescribed velocity. Using a numerical method, we obtain the nonsteady aerodynamic characteristics of the body for different accelerations. An analytical method is proposed for calculating the pressure distribution at the initial moment of time and the maximum forces present in the case of sudden motion.

1. Formulation of the Problem and Method of Numerical Solution. Let a solid of revolution of a specified form begin to move from a state of rest at the initial moment of time $t = 0$. Moving with steady acceleraton during the time T, the body is assumed to reach a velocity corresponding to a prescribed Mach number M. The gas is considered to be ideal and to be in a state of rest with a constant pressure p_0 and density ρ_0. The adiabatic exponent of the gas is $\gamma = 1.4$.

In the coordinate system connected with the body, the flow of the gas is described by the two-dimensional nonsteady Euler equations

$$\frac{\partial}{\partial t}(\rho y) + \frac{\partial}{\partial x}(\rho u y) + \frac{\partial}{\partial y}(\rho v y) = 0,$$

$$\frac{\partial}{\partial t}(\rho u y) + \frac{\partial}{\partial x}[(p + \rho u^2) y] + \frac{\partial}{\partial y}(\rho u v y) = 0,$$

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 83-88, May-June, 1989. Original article submitted September 3, 1987; revision submitted January 22, 1988.